Astrodynamic Pedantry—Earth's Gravitational Parameter, Equatorial Radius, and Angular Velocity

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Abstract

A great deal of confusion revolves around the earth's gravitational parameter, equatorial radius, and angular velocity. Experience with source code from numerous high, medium, and low fidelity astrodynamics related modeling tools clearly demonstrates this quandary. Confusion is understandable given past approaches assumed only a single value for each constant. In addition, references that precisely define these values are often difficult to interpret. While the details covered here may be pedantic in nature, the devil is quite often in the details when it comes to modeling astrodynamics related problems.

1 Overview

The 2$^{\text{nd}}$ edition of the WGS 84 standard [2] defines the modern oblate spheroid semimajor axis used to convert between Cartesian and geodetic coordinates. This edition also describes the concept of a nominal mean earth angular velocity along with two additional definitions intended for high precision modeling and simulation. One version is intended to compensate for the Coriolis effect (and centripetal acceleration) when converting velocity (and acceleration) vectors between earth centered earth fixed (ECEF) and earth centered inertial (ECI) reference frames. Another version of the angular velocity is used to compute the earth's angular rotation relative to a precessing reference frame. This rate is used when the sidereal time at an epoch, which is relative to an equinox-based ECI reference frame, is given and a new value is desired at a different time.

The 3$^{\text{rd}}$ edition of the WGS 84 standard [3] adopts the EGM96 gravity model while continuing to use the previous WGS 84 ellipsoid definition. This update introduces the need for two semimajor axis values defining the earth. One is used for Cartesian/geodetic coordinate conversions while the other is used as the scaling factor when evaluating the EGM96 gravity model.

The International Terrestrial Reference Frame (ITRF) is defined in the *IERS Conventions (2010)*, a.k.a. *Technical Note 36* [4], or simply *TN 36*. This is the modern ECEF reference frame used for precision astrodynamics work. It is located relative to inertial space as a Cartesian coordinate system. TN 36 also specifies an oblate spheroid model to be used for conversions between ITRF Cartesian and geodetic coordinates. While the equatorial radius of the earth is the same as the WGS 84 model, the flattening factor differs slightly.

2 Transmission of Coordinates

The well-known computer scientist Dr. Tanenbaum penned [6, p. 254], "The nice thing about standards is that you have so many to choose from." The field of astrodynamics eagerly illustrates this concept. With respect to ECI reference frames, one can choose from the classic true equator mean equinox (TEME), true equator true equinox (TETE), J2000, and modern International Astronomical Union (IAU) Geocentric Celestial Reference Frame (GCRF) definitions, just to name a few. Geodetic reference frames are at least essentially limited to either the WGS 84 standard or the IAU adopted GRS80 model.

As described in the overview §1, the modern Cartesian ECEF reference frame is the ITRF. For all practical purposes, no other Cartesian ECEF reference frame is defined. Instead, differences from the ITRF exist due to an application's omissions from the full coordinate transformation theory (e.g., not including polar motion, not modeling seasonal variations when interpolating earth orientation parameters (EOP), etc.) or simplifications to models that indirectly affect realization of the ITRF (e.g., a spherical earth model where the difference between the geocentric and geodetic latitudes is ignored). In addition, the"differences between WGS 84 and ITRF are in the centimeter range worldwide." [3, p. 7-1]

To eliminate unnecessary errors caused by differences in reference frame and coordinate transformation theories, it is recommended state vectors (position and velocity) are always communicated via Cartesian ECEF coordinates. Such coordinates allow for the state vectors to to be represented by the greatest level of fidelity supported by the source application algorithms. The use of ECI or geodetic coordinates when transferring state vectors may introduce errors that could otherwise easily be avoided.

Consider a high precision application making use of the GCRF when propagating ephemeris while incorporating the full modern reduction theories (ECEF to ECI coordinate transformations) defined by the IAU. If state vectors are relayed via ECEF coordinates, then a lower fidelity application can convert these vectors to an internal *computational reference frame* where the equations of motion can be propagated in accordance with the application's inherent level of fidelity. In contrast, if state vectors are transmitted via the modern GCRF definition and the recipient application only understands a simplified (but very useful) TEME reference frame, a significant error will immediately be introduced. Accurate transformations to an ECEF or another ECI reference frame will be impossible because the less sophisticated application will have no concept of what the GCRF reference frame is.

3 Cartesian ECEF vs. Geodetic Coordinates

As of this writing, the 3$^{\mathrm{rd}}$ edition of the WGS 84 standard is typically considered to be the authoritative model for transformations between Cartesian and geodetic (latitude, longitude, altitude) coordinates. It is the DoD standard, including its adoption by the Navstar Global Positioning System (GPS) [1]. Table 3.1 of the WGS 84 standard [3, p. 3-5] defines the earth's semimajor axis and the reciprocal of flattening:

$$\begin{aligned} a &= 6378137.0 \text{ meters} \\ 1/f &= 298.257223563 \end{aligned}$$

TN 36 references the GRS80 ellipsoid model for conversions between ECEF Cartesian and "geographical coordinates" [4, §4.2.6]. This model adopts the same semimajor axis value while employing an ellipsoid inverse flattening of $1/f = 298.257222101$. Unlike the GRS80 model, the WGS 84 flattening is derived from a truncated form of "the normalized second degree zonal harmonic gravitational coefficient…" [3, p. 7-2] This results in a maximum discrepancy of just over a tenth of a millimeter between the two definitions when converting between geodetic and Cartesian coordinates on the surface of the earth.

Keep in mind, the ellipsoid definition is just a mathematical construct. It is an agreed upon standard used to define latitude and longitude given ECEF Cartesian coordinates (just as the location of the north pole is defined by convention and does not align with the true spin axis of the earth, which is always changing). The concept of a *true* earth radius and flattening are irrelevant as long as these model parameters are representative of an ellipsoidal approximation of the earth and are used consistently.

4 Gravitational Model

The EGM96 and EGM2008 terrestrial time (TT) compatible gravitational parameter [4, p. 79] is

$$GM_{\oplus} = 398600.4415 \ \frac{\text{km}^3}{\text{sec}^2}$$

It is often confused with the geocentric coordinate time (TCG) compatible value of 398600.4418 km^3/sec^2. This is most likely because Table 1.1 of TN 36 [4, p. 18] lists the TCG value while the fine print specifies the TT version. To aid in clarifying which value should be used when evaluating either gravitational model, consult the *EGM2008 README_FIRST.pdf* document available at the National Geospatial-Intelligence Agency (NGA) website or the EGM96 *readme.egm96* text file available via NASA's Crustal Dynamics Data Information System (CDDIS) ftp server. These sites contain the EGM2008 and EGM96 gravitational models along with documentation (somewhat) clearly explaining their use.

TT is a time scale realized at sea level on the earth. In contrast, TCG is realized such that it is not affected by relativistic gravitational effects—i.e., a location infinitely far from the earth. Even though TCG is intended to be used as the time scale for modeling the motion of celestial bodies, TT is used in practice. As explained by Vallado [7, p. 194], TCG may be inconvenient because it "is a coordinate time representing the independent argument of the equations of motion of bodies in its frame and will not be ordinarily kept by any physically real clock" while TT is more convenient since "it is realized via TAI and UTC." Earth precession and nutation formulas illustrate the use of TT as the independent parameter. Algorithms predicting the orbits of the sun and moon relative to the earth are also typically developed as a function of TT.

The gravitational scaling radius associated with the EGM96 and EGM2008 models is denoted in the *readme* documentation provided with each model and in the TN 36 fine print [4, p. 79]. Note that this differs from the semimajor axis value used to define the earth's oblate spheroid model:

$$R_{\oplus} = 6378.1363 \text{ km}$$

The gravitational parameter and the associated scaling factor are the primary values defining modern gravitational potential models. There is little reason to include high order spherical harmonics if the wrong first order modeling parameters are employed.

5 Angular Velocity

The WGS 84 and GRS80 ellipsoid definitions include a *nominal* mean angular velocity of the earth with respect to inertial space [3, p. 3-4] [4, p. 19.]:

$$\omega_{\oplus} = 7292115 \times 10^{-11} \ \frac{\text{rad}}{\text{sec}}$$

High fidelity applications must employ a more precise definition of the earth's angular velocity. A distinction needs to be made between the angular velocity of the earth relative to inertial space and the angular velocity relative to a precessing reference frame.

When transforming velocity between ECEF and ECI reference frames, the Coriolis effect must be taken into account. Centripetal acceleration must also be incorporated when transforming acceleration. Both corrections involve cross products between the satellite's position vector and the earth's angular velocity vector. To convert the reference frame in which the velocity and acceleration derivatives are taken, the angular velocity of the earth w.r.t. inertial space, 7292115×10^{-11} rad/sec, or the higher fidelity [7, p. 222]

$$\omega_{\oplus} = 7.292115146706979 \times 10^{-5} \left(1 - \frac{\text{LOD}}{86400}\right) \frac{\text{rad}}{\text{sec}}$$

should be used, where LOD is the EOP value Length of Day. Seidelmann [5, p. 51] presents this formula to 11 significant figures in the 1992 version of the *Explanatory Supplement to the Astronomical Almanac*. LOD is [7, p. 222] "the instantaneous rate of change (in seconds) of UT1 with respect to a uniform time scale (UTC or TAI)." Inspection of the above equation and taking note that the maximum magnitude of LOD is on the order of 4 milliseconds, setting LOD to zero when it is unknown results in a more accurate approximation of $\omega_\oplus$ than using the truncated version 7292115×10^{-11} rad/sec.

Sidereal time is the earth rotation angle measured relative to an equinox-based ECI reference frame. Such a reference frame is realized by the slowly drifting intersection of the earth's equator and the ecliptic, which form the ECI x-axis. Vallado [7, p. 207] explains the "...motion of the ecliptic plane due to precession causes the equinox to move along [in the plane of] the equator...", which affects the angular velocity of the earth relative to this precessing frame. This comes into play when computing the sidereal time relative to a known value at some epoch. For example,

$$\theta_{GMST} = \theta_{GMST_0} + \omega_{\oplus_{prec}} \Delta_t$$

where θ_{GMST_0} is a known Greenwich mean sidereal time (GMST) at some epoch and Δ_t is the change in time since that epoch. The formula for computing this form of the earth's angular velocity is [7, p. 180] [5, p. 52],

$$\omega_{\oplus_{prec}} = 1.002737909350795 + 5.9006 \times 10^{-11} T_{UT1} - 5.9 \times 10^{-15} T_{UT1}^2 \ \frac{\text{rev}}{\text{day}}$$

where T_{UT1} is the number of Julian centuries in UT1,

$$T_{UT1} = \frac{JD_{UT1} - 2,451,545_{UT1}}{36,525}$$

and one rev per day is 2π radians per 86400 seconds. This polynomial is derived by differentiation of sidereal time θ_{GMST} [5, p. 51]. Beware of software where the earth rotation rate has been set to the radians per second equivalent of the constant in the above equation, $7.292115855306587 \times 10^{-5}$ rad/sec.

References

[1] ARINC Engineering Services, LLC, Navstar GPS Space Segment/Navigation User Interfaces, Tech. Rep. IS-GPS-200, Navstar GPS Joint Program Office, El Segundo, CA, March 2006.

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[4] G. Petit and B. Luzum, *IERS Conventions (2010)*, IERS Technical Note No. 36, Verlag des Bundesamts für Kartographie und Geodäsie, Frankfurt am Main, 2010.

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[7] D. A. Vallado, *Fundamentals of Astrodynamics and Applications*, Microcosm Press, Hawthorne, CA, forth ed., 2013.